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Modal analysis of asymmetric rotor system with isotropic stator using modulated coordinates

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Abstract

A new modal analysis method for rotor systems with periodically time-varying parameters is proposed. The essence of the method is to introduce the modulated coordinates such that the periodically time-varying linear differential equations can be effectively transformed to the equivalent time-invariant linear differential equations. This paper presents the modal analysis procedure for the asymmetric rotor system, of which rotating and stationary parts possess asymmetric and isotropic properties, respectively. With a simple asymmetric rotor model, the analytical modal analysis procedure is illustrated and compared with the conventional method based on formulation in the rotating coordinates. A numerical example with a flexible asymmetric rotor model is also provided to demonstrate the effectiveness of the proposed method.

1. Introduction

The presence of asymmetric properties, either in the rotor part or in the stator part, can significantly affect the dynamic characteristics of a rotor system. However, there are few modal analysis methods found to be appropriate for rotating machinery with asymmetric properties. In particular, modal analysis of asymmetric rotor systems, when the equations of motion are written

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in the stationary coordinates, becomes mathematically involved, since the system matrices of governing linear differential equations contain periodically time-varying parameters, leading to Hill's equation [1]. A direct method has been introduced in Ref. [2] to perform a modal analysis for such periodically time-varying parameter systems by employing the time-varying eigenvectors. However, the method, up until now, is limited to the systems that are represented in the real coordinates. The analysis of time-varying parameter systems has often adopted the harmonic balance method, which has been usually confined to the analysis of natural frequencies [3].

The present study proposes a new modal analysis method for rotor systems with asymmetric properties employing modulated coordinates. This paper deals with *asymmetric rotors*, which possess asymmetric properties only in the rotor part. For asymmetric rotor systems, the periodically time-varying linear differential equations expressed in the stationary coordinates can be transformed to the time-invariant linear differential equations expressed in the rotating coordinates. Then the modal analysis method becomes essentially the same as the ordinary complex modal analysis method developed for *anisotropic rotors*, which possess asymmetric properties only in the stator part [4,5]. However, it is often required that the modal analysis formulated in the rotating coordinates be transformed back into the stationary coordinates, in order to develop the modal testing schemes normally carried out in the stationary coordinates.

To resolve such a difficulty, Lee et al. [6–8] proposed an idea of complex modal analysis using a coordinate modulation technique, which makes it possible to take advantage of excitations and measurements based on the stationary coordinates. On the other hand, they introduced a concept of normal and reverse directional frequency response functions for detection of asymmetry in rotors. The reverse-directional frequency response function (r-dFRF) is found to be a sensitive indicator of the asymmetric, deviatoric property of the rotor system whereas the normal-directional frequency response function (n-dFRF) is an indicator of the symmetric, mean property of the rotor system. However, they discussed neither the eigenvalues nor the eigenvectors related to the modulated coordinates.

The primary aim of this paper is to introduce a generalized theory of complex modal analysis method using the modulated complex stationary coordinates for asymmetric rotor systems with isotropic stators. In this case, the proposed modal method provides a complete modal solution such as eigenvalues and latent vectors (eigenvectors) for asymmetric rotor systems. The method introduces the modulated complex stationary coordinates to derive an equivalent linear time-invariant equation of motion. Then, the characteristics of eigenvalues and latent vectors are theoretically investigated thoroughly by using the equivalent linear time-invariant equation of motion. Two analytical and numerical examples are treated to demonstrate the effectiveness of the proposed method.

2. Modal analysis of asymmetric rotor system

2.1. Equation of motion in the modulated stationary coordinates

The equation of motion of asymmetric rotor system can be written, using the complex stationary coordinates, as [3,6]

$$\mathbf{M}_f\ddot{\mathbf{p}}(t) + \mathbf{M}_r\ddot{\bar{\mathbf{p}}}(t)\mathrm{e}^{\mathrm{j}2\Omega t} + \mathbf{C}_f\dot{\mathbf{p}}(t) + \mathbf{C}_r\dot{\bar{\mathbf{p}}}(t)\mathrm{e}^{\mathrm{j}2\Omega t} + \mathbf{K}_f\mathbf{p}(t) + \mathbf{K}_r\bar{\mathbf{p}}(t)\mathrm{e}^{\mathrm{j}2\Omega t} = \mathbf{g}(t) \tag{1a}$$

or

$$\begin{bmatrix} \mathbf{M}_f & \mathbf{M}_r \\ \bar{\mathbf{M}}_r & \bar{\mathbf{M}}_f \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{p}}(t) \\ \ddot{\bar{\mathbf{p}}}(t)\mathrm{e}^{\mathrm{j}2\Omega t} \end{Bmatrix} + \begin{bmatrix} \mathbf{C}_f & \mathbf{C}_r \\ \bar{\mathbf{C}}_r & \bar{\mathbf{C}}_f \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{p}}(t) \\ \dot{\bar{\mathbf{p}}}(t)\mathrm{e}^{\mathrm{j}2\Omega t} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_f & \mathbf{K}_r \\ \bar{\mathbf{K}}_r & \bar{\mathbf{K}}_f \end{bmatrix} \begin{Bmatrix} \mathbf{p}(t) \\ \bar{\mathbf{p}}(t)\mathrm{e}^{\mathrm{j}2\Omega t} \end{Bmatrix} = \begin{Bmatrix} \mathbf{g}(t) \\ \bar{\mathbf{g}}(t)\mathrm{e}^{\mathrm{j}2\Omega t} \end{Bmatrix}, \tag{1b}$$

where $\mathbf{M}_i$, $\mathbf{C}_i$ and $\mathbf{K}_i$ denote the complex valued $N \times N$ generalized mass, damping and stiffness matrices, respectively, the subscripts $i = f, r$ refer to the mean and deviatoric values, respectively; $\mathbf{p}(t) = \mathbf{y}(t) + \mathrm{j}\mathbf{z}(t)$ and $\mathbf{g}(t) = \mathbf{f}_\mathbf{y}(t) + \mathrm{j}\mathbf{f}_\mathbf{z}(t)$ are the $N \times 1$ complex response and input vectors, respectively, $\mathbf{y}(t)$ and $\mathbf{z}(t)$ are the real valued response vectors, and, $\mathbf{f}_\mathbf{y}(t)$ and $\mathbf{f}_\mathbf{z}(t)$ are the real valued input vectors, in the direction of Y and Z in the stationary coordinates, forming a plane perpendicular to the bearing axis, respectively, Ω is the rotational speed of the shaft, N is the dimension of the complex coordinate vector, j is the imaginary number; the bar indicates the complex conjugate. Note here that the time-varying parameters associated with the harmonic frequency of twice the rotational speed appear due to the asymmetry in the rotor part. Note that the system matrices, including the effect of the gyroscopic moment, internal damping and fluid-film bearing characteristics, may be dependent upon the rotational speed. However, they become constant for given rotational speed.

It will prove convenient to introduce the modulated complex coordinate and force vectors, $\tilde{\mathbf{p}}(t)$ and $\tilde{\mathbf{g}}(t)$, defined as

$$\tilde{\mathbf{p}}(t) = \bar{\mathbf{p}}(t)\mathrm{e}^{\mathrm{j}2\Omega t}, \quad \tilde{\mathbf{g}}(t) = \bar{\mathbf{g}}(t)\mathrm{e}^{\mathrm{j}2\Omega t}. \tag{2}$$

Substituting relations (2) into Eq. (1b), we obtain

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t), \tag{3}$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_f & \mathbf{M}_r \\ \bar{\mathbf{M}}_r & \bar{\mathbf{M}}_f \end{bmatrix}_{2N\times 2N}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_f & \mathbf{C}_r - \mathrm{j}4\Omega\mathbf{M}_r \\ \bar{\mathbf{C}}_r & \bar{\mathbf{C}}_f - \mathrm{j}4\Omega\bar{\mathbf{M}}_f \end{bmatrix}_{2N\times 2N},$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_f & \mathbf{K}_r - \mathrm{j}2\Omega\mathbf{C}_r - 4\Omega^2\mathbf{M}_r \\ \bar{\mathbf{K}}_r & \bar{\mathbf{K}}_f - \mathrm{j}2\Omega\bar{\mathbf{C}}_f - 4\Omega^2\bar{\mathbf{M}}_f \end{bmatrix}_{2N\times 2N},$$

$$\mathbf{q}(t) = \begin{Bmatrix} \mathbf{p}(t) \\ \tilde{\mathbf{p}}(t) \end{Bmatrix}_{2N\times 1}, \quad \mathbf{f}(t) = \begin{Bmatrix} \mathbf{g}(t) \\ \tilde{\mathbf{g}}(t) \end{Bmatrix}_{2N\times 1}.$$

Note here that the periodically time-varying nature of the original system parameters is all taken by the newly defined, modulated stationary coordinates, leading to the time-invariant parameter, but speed-dependent, differential equation (3) defined with respect to an unusual, but still stationary, coordinate system.

2.2. Modal analysis

Assuming the solution form of $\mathbf{q}(t) = \mathbf{u}_c e^{\lambda t}$, one obtains the sets of homogeneous equations associated with Eq. (3) as [4]

$$\mathbf{D}(\lambda_r^i)\mathbf{u}_{cr}^i = \mathbf{0} \quad \text{and} \quad \bar{\mathbf{v}}_{cr}^{i\mathrm{T}}\mathbf{D}(\lambda_r^i) = \mathbf{0}^{\mathrm{T}}, \quad r = \pm 1, \pm 2, \ldots, \pm N, \quad i = B, F, \tag{4a}$$

where the lambda matrix of degree two is given by

$$\mathbf{D}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K} = \begin{bmatrix} \mathbf{D}_f(\lambda) & \tilde{\mathbf{D}}_r(\lambda) \\ \mathbf{D}_r(\lambda) & \tilde{\mathbf{D}}_f(\lambda) \end{bmatrix}, \tag{4b}$$

with

$$\begin{aligned} \mathbf{D}_f(\lambda) &= \lambda^2\mathbf{M}_f + \lambda\mathbf{C}_f + \mathbf{K}_f, \\ \mathbf{D}_r(\lambda) &= \lambda^2\bar{\mathbf{M}}_r + \lambda\bar{\mathbf{C}}_r + \bar{\mathbf{K}}_r, \\ \tilde{\mathbf{D}}_r(\lambda) &= \lambda^2\mathbf{M}_r + \lambda(\mathbf{C}_r - \mathrm{j}4\Omega\mathbf{M}_r) + \mathbf{K}_r - \mathrm{j}2\Omega\mathbf{C}_r - 4\Omega^2\mathbf{M}_r, \\ \tilde{\mathbf{D}}_f(\lambda) &= \lambda^2\bar{\mathbf{M}}_f + \lambda(\bar{\mathbf{C}}_f - \mathrm{j}4\Omega\bar{\mathbf{M}}_f) + \bar{\mathbf{K}}_f - \mathrm{j}2\Omega\bar{\mathbf{C}}_f - 4\Omega^2\bar{\mathbf{M}}_f, \end{aligned} \tag{4c}$$

and the right and left latent vectors take the form of

$$\mathbf{u}_c = \begin{Bmatrix} \mathbf{u} \\ \breve{\mathbf{u}} \end{Bmatrix}, \quad \bar{\mathbf{v}}_c = \begin{Bmatrix} \bar{\mathbf{v}} \\ \bar{\breve{\mathbf{v}}} \end{Bmatrix}. \tag{4d}$$

The latent roots (eigenvalues) λ are determined from the characteristic polynomial of order $4N$

$$|\mathbf{D}(\lambda)| = \begin{vmatrix} \mathbf{D}_f(\lambda) & \tilde{\mathbf{D}}_r(\lambda) \\ \mathbf{D}_r(\lambda) & \tilde{\mathbf{D}}_f(\lambda) \end{vmatrix} = 0. \tag{5}$$

Here, the pair of eigenvalues, equal in subscript value but different in sign of subscript, are dependent upon each other; they are complex conjugate pairs with a shift parameter $\mathrm{j}2\Omega$, as will be shown later. And the superscripts B and F implicitly refer to the backward and forward modes, respectively [4].

The latent vectors, obtained from Eq. (4a), are normalized so as to satisfy the bi-orthonormality condition given by

$$(\lambda_r^i + \lambda_s^k)\bar{\mathbf{v}}_{cs}^{k\mathrm{T}}\mathbf{M}\mathbf{U}_{cr}^i + \bar{\mathbf{v}}_{cs}^{k\mathrm{T}}\mathbf{C}\mathbf{u}_{cr}^i = \delta_{sr}^{ki}, \quad r, s = +1, \ldots \pm N; \quad i, k = B, F \tag{6a}$$

or, for $r = s, i = k$,

$$\bar{\mathbf{v}}_c^{\mathrm{T}}[\mathbf{D}'(\lambda)]\mathbf{u}_c = \begin{Bmatrix} \bar{\mathbf{v}}^{\mathrm{T}} & \bar{\breve{\mathbf{v}}}^{\mathrm{T}} \end{Bmatrix} \begin{bmatrix} \mathbf{D}_f'(\lambda) & \tilde{\mathbf{D}}_r'(\lambda) \\ \mathbf{D}_r'(\lambda) & \tilde{\mathbf{D}}_f'(\lambda) \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \breve{\mathbf{u}} \end{Bmatrix} = 1, \tag{6b}$$

where

$$\mathbf{D}'(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbf{D}(\lambda) = 2\lambda\mathbf{M} + \mathbf{C}, \tag{6c}$$

and the Kronecker delta is defined as

$$\delta_{sr}^{ki} = \begin{cases} 1 & \text{when } i = k \text{ and } r = s, \\ 0 & \text{otherwise.} \end{cases} \tag{6d}$$

Since the eigensolution takes the form of

$$\mathbf{q}(t) = \begin{Bmatrix} \mathbf{p}(t) \\ \tilde{\mathbf{p}}(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}(t) \\ \bar{\mathbf{p}}(t)\mathrm{e}^{\mathrm{j}2\Omega t} \end{Bmatrix} = \begin{Bmatrix} \mathbf{u} \\ \breve{\mathbf{u}} \end{Bmatrix} \mathrm{e}^{\lambda t}, \tag{7a}$$

it holds, for each eigensolution,

$$\begin{gathered} \mathbf{p}(t) = \mathbf{u}_r^i \mathrm{e}^{\lambda_r^i t}, \\ \tilde{\mathbf{p}}(t) = \bar{\mathbf{p}}(t)\mathrm{e}^{\mathrm{j}2\Omega t} = \bar{\mathbf{u}}_s^k \mathrm{e}^{(\bar{\lambda}_s^k + \mathrm{j}2\Omega)t} = \breve{\mathbf{u}}_r^i \mathrm{e}^{\lambda_r^i t}. \end{gathered} \tag{7b}$$

or, equivalently,

$$\bar{\mathbf{u}}_s^k = \breve{\mathbf{u}}_r^i \quad \text{and} \quad \bar{\lambda}_s^k + \mathrm{j}2\Omega = \lambda_r^i. \tag{7c}$$

In order to satisfy the above two relations (7c), it should hold

$$s = -r, \quad i = k \quad \text{or} \quad s = r, \quad i = k, \ \mathrm{Im}\left[\lambda_r^i\right] = \Omega. \tag{8}$$

Note that the second set of conditions is not met in general for all Ω. From the first set of conditions, it can be shown that the eigenvalues and right latent vectors, which are associated with the pair of positive and negative subscripts, satisfy the relations given by

$$\lambda_r^i, \ \lambda_{-r}^i (= \bar{\lambda}_r^i + \mathrm{j}2\Omega), \tag{9a}$$

$$\mathbf{u}_{cr}^i = \begin{Bmatrix} \mathbf{u} \\ \breve{\mathbf{u}} \end{Bmatrix}_r^i = \begin{Bmatrix} \mathbf{u}_r \\ \bar{\mathbf{u}}_{-r} \end{Bmatrix}^i, \quad \mathbf{u}_{c-r}^i = \begin{Bmatrix} \mathbf{u} \\ \breve{\mathbf{u}} \end{Bmatrix}_{-r}^i = \begin{Bmatrix} \bar{\breve{\mathbf{u}}} \\ \bar{\mathbf{u}} \end{Bmatrix}_r^i = \begin{Bmatrix} \mathbf{u}_{-r} \\ \bar{\mathbf{u}}_r \end{Bmatrix}^i, \tag{9b}$$

$$r = \pm 1, \pm 2, \ldots \pm N, \quad i = B, F.$$

Similarly, one can obtain the relation between the left latent vectors as

$$\mathbf{v}_{cr}^i = \begin{Bmatrix} \mathbf{v} \\ \breve{\mathbf{v}} \end{Bmatrix}_r^i = \begin{Bmatrix} \mathbf{v}_r \\ \bar{\mathbf{v}}_{-r} \end{Bmatrix}^i, \quad \mathbf{v}_{c-r}^i = \begin{Bmatrix} \mathbf{v} \\ \breve{\mathbf{v}} \end{Bmatrix}_{-r}^i = \begin{Bmatrix} \bar{\breve{\mathbf{v}}} \\ \bar{\mathbf{v}} \end{Bmatrix}_r^i = \begin{Bmatrix} \mathbf{v}_{-r} \\ \bar{\mathbf{v}}_r \end{Bmatrix}^i, \tag{10}$$

$$r = \pm 1, \pm 2, \ldots \pm N, \quad i = B, F.$$

Note that the eigenvalue λ_{-r}^i can be derived as the complex conjugate of λ_r^i with a shift of $\mathrm{j}2\Omega$ and that the corresponding latent vectors can also be derived from each other as given in Eqs. (9b) and (10).

The complex response vector $\mathbf{p}(t)$ can then be expanded in terms of the right latent vectors as

$$\mathbf{p}(t) = \sum_{i=B,F} \sum_{r=-N}^{N}{}' \{\mathbf{u}_r^i \eta_r^i(t)\}, \tag{11a}$$

where the principal coordinates $\eta_r^i(t)$ satisfy the 4N sets of modal equations given by

$$\dot{\eta}_r^i(t) - \lambda_r^i \eta_r^i(t) = \bar{\mathbf{v}}_r^{i\mathrm{T}} \mathbf{g}(t) + \bar{\tilde{\mathbf{v}}}_r^{i\mathrm{T}} \tilde{\mathbf{g}}(t), \quad r = \pm 1, \pm 2, \ldots \pm N, \ i = B, F. \tag{11b}$$

Here, $\sum_{r=-N}^{N\prime}$ is the summation operator excluding $r = 0$. From Eq. (11), we can derive the input-output relation in the frequency domain as

$$\mathbf{P}(\mathrm{j}\omega) = \begin{bmatrix} \mathbf{H}_{gp} & \mathbf{H}_{\tilde{g}p} \end{bmatrix} \begin{Bmatrix} \mathbf{G}(\mathrm{j}\omega) \\ \tilde{\mathbf{G}}(\mathrm{j}\omega) \end{Bmatrix}, \tag{12a}$$

where

$$\mathbf{H}_{gp}(\mathrm{j}\omega) = \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}\bar{\mathbf{v}}^{\mathrm{T}}}{\mathrm{j}\omega - \lambda} \right]_r^i = \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{\mathbf{u}_r^i \bar{\mathbf{v}}_{-r}^{i\mathrm{T}}}{\mathrm{j}\omega - \lambda_r^i} + \frac{\mathbf{u}_{-r}^i \bar{\mathbf{v}}_r^{i\mathrm{T}}}{\mathrm{j}\omega - \lambda_{-r}^i} \right],$$
$$\mathbf{H}_{\tilde{g}p}(\mathrm{j}\omega) = \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}\bar{\tilde{\mathbf{v}}}^{\mathrm{T}}}{\mathrm{j}\omega - \lambda} \right]_r^i = \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{\mathbf{u}_r^i \mathbf{v}_r^{i\mathrm{T}}}{\mathrm{j}\omega - \lambda_r^i} + \frac{\mathbf{u}_{-r}^i \mathbf{v}_r^{i\mathrm{T}}}{\mathrm{j}\omega - \lambda_{-r}^i} \right], \tag{12b}$$
$$\tilde{\mathbf{G}}(\mathrm{j}\omega) = \hat{\mathbf{G}}\{\mathrm{j}(\omega - 2\Omega)\}.$$

Here $\mathbf{P}(\mathrm{j}\omega)$, $\mathbf{G}(\mathrm{j}\omega)$ and $\hat{\mathbf{G}}(\mathrm{j}\omega)$ are the Fourier transforms of $\mathbf{p}(t)$, $\mathbf{g}(t)$ and $\bar{\mathbf{g}}(t)$, respectively, and, $\mathbf{H}_{gp}$ and $\mathbf{H}_{\tilde{g}p}$ are referred to as the normal directional frequency response matrix (n-dFRM) and the reverse directional frequency response matrix (r-dFRM), respectively.

2.3. Norm of residue matrices for weakly asymmetric rotor

The homogeneous Eq. (4) can be rewritten, for the weakly asymmetric rotor with a small perturbation ε, the degree of rotating asymmetry, as

$$\begin{bmatrix} \mathbf{D}_f(\lambda_r^i) & \varepsilon\tilde{\mathbf{D}}_r(\lambda_r^i) \\ \varepsilon\mathbf{D}_r(\lambda_r^i) & \tilde{\mathbf{D}}_f(\lambda_r^i) \end{bmatrix} \begin{Bmatrix} \mathbf{u}_r^i \\ \breve{\mathbf{u}}_r^i \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{13a}$$

$$\begin{Bmatrix} \bar{\mathbf{v}}_r^{i\mathrm{T}} & \bar{\tilde{\mathbf{v}}}_r^{i\mathrm{T}} \end{Bmatrix} \begin{bmatrix} \mathbf{D}_f(\lambda_r^i) & \varepsilon\tilde{\mathbf{D}}_r(\lambda_r^i) \\ \varepsilon\mathbf{D}_r(\lambda_r^i) & \tilde{\mathbf{D}}_f(\lambda_r^i) \end{bmatrix} = \begin{Bmatrix} \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \end{Bmatrix}, \tag{13b}$$

where the terms preceded by ε indicate the small perturbations. When the rotor becomes isotropic, i.e. $\varepsilon = 0$, Eq. (13) reduces to

$$\begin{Bmatrix} \mathbf{D}_f(\lambda_{r0}^i)\mathbf{u}_{r0}^i \\ \tilde{\mathbf{D}}_f(\lambda_{r0}^i)\breve{\mathbf{u}}_{r0}^i \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{14a}$$

$$\begin{Bmatrix} \bar{\mathbf{v}}_{r0}^{i\mathrm{T}} \mathbf{D}_f(\lambda_{r0}^i) & \bar{\tilde{\mathbf{v}}}_{r0}^{i\mathrm{T}} \tilde{\mathbf{D}}_f(\lambda_{r0}^i) \end{Bmatrix} = \begin{Bmatrix} \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \end{Bmatrix}, \tag{14b}$$

with the latent roots satisfying

$$\left|\mathbf{D}_f(\lambda_{r0}^i)\right| = \left|\tilde{\mathbf{D}}_f(\lambda_{-r0}^i)\right| = 0, \quad r = 1, 2, \ldots, N, \ \ i = B, F, \tag{14c}$$

where the subscript '0' means the associated isotropic rotor with $\varepsilon = 0$. Note that, according to the notational conventions (10) and (14c), we have

$$\begin{aligned} \mathbf{u}_{r0}^i &= \bar{\mathbf{v}}_{r0}^i = \mathbf{0} \quad \text{for } r<0, \\ \breve{\mathbf{u}}_{r0}^i &= \bar{\breve{\mathbf{v}}}_{r0}^i = \mathbf{0} \quad \text{for } r>0, \end{aligned} \tag{15}$$

which are consistent with the results in Eqs. (9) and (10).

The latent vectors in Eq. (13) reduce to, for $r>0$,

$$\begin{Bmatrix} \mathbf{u}_r^i \\ \breve{\mathbf{u}}_r^i \end{Bmatrix} \cong \begin{Bmatrix} \mathbf{u}_{r0}^i \\ \breve{\mathbf{u}}_{r0}^i \end{Bmatrix} + \varepsilon \begin{Bmatrix} \mathbf{u}_{r1}^i \\ \breve{\mathbf{u}}_{r1}^i \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_{r0}^i + \varepsilon \mathbf{u}_{r1}^i \\ \varepsilon \breve{\mathbf{u}}_{r1}^i \end{Bmatrix}, \tag{16a}$$

$$\begin{Bmatrix} \bar{\mathbf{v}}_r^{iT} & \bar{\breve{\mathbf{v}}}_r^{iT} \end{Bmatrix} \cong \begin{Bmatrix} \bar{\mathbf{v}}_{r0}^{iT} & \bar{\breve{\mathbf{v}}}_{r0}^{iT} \end{Bmatrix} + \varepsilon \begin{Bmatrix} \bar{\mathbf{v}}_{r1}^{iT} & \bar{\breve{\mathbf{v}}}_{r1}^{iT} \end{Bmatrix} = \begin{Bmatrix} \bar{\mathbf{v}}_{r0}^{iT} + \varepsilon \bar{\mathbf{v}}_{r1}^{iT} & \varepsilon \bar{\breve{\mathbf{v}}}_{r1}^{iT} \end{Bmatrix}, \tag{16b}$$

and, for $r<0$,

$$\begin{Bmatrix} \mathbf{u}_r^i \\ \breve{\mathbf{u}}_r^i \end{Bmatrix} \cong \begin{Bmatrix} \mathbf{u}_{r0}^i \\ \breve{\mathbf{u}}_{r0}^i \end{Bmatrix} + \varepsilon \begin{Bmatrix} \mathbf{u}_{r1}^i \\ \breve{\mathbf{u}}_{r1}^i \end{Bmatrix} = \begin{Bmatrix} \varepsilon \mathbf{u}_{r1}^i \\ \breve{\mathbf{u}}_{r0}^i + \varepsilon \breve{\mathbf{u}}_{r1}^i \end{Bmatrix}, \tag{16c}$$

$$\begin{Bmatrix} \bar{\mathbf{v}}_r^{iT} & \bar{\breve{\mathbf{v}}}_r^{iT} \end{Bmatrix} \cong \begin{Bmatrix} \bar{\mathbf{v}}_{r0}^{iT} & \bar{\breve{\mathbf{v}}}_{r0}^{iT} \end{Bmatrix} + \varepsilon \begin{Bmatrix} \bar{\mathbf{v}}_{r1}^{iT} & \bar{\breve{\mathbf{v}}}_{r1}^{iT} \end{Bmatrix} = \begin{Bmatrix} \varepsilon \bar{\mathbf{v}}_{r1}^{iT} & \bar{\breve{\mathbf{v}}}_{r0}^{iT} + \varepsilon \bar{\breve{\mathbf{v}}}_{r1}^{iT} \end{Bmatrix}. \tag{16d}$$

On the other hand, the latent roots, due to the presence of asymmetry ε in the rotor model, can be approximated as

$$\lambda_r^i \cong \lambda_{r0}^i + \varepsilon^2 \lambda_{r2}^i, \quad r = \pm 1, \pm 2, \ldots, \pm N, \tag{17}$$

of which proof is given in Appendix. We can now express the norm order of the residue matrices in Eq. (12b), from Eq. (16), as [9]

$$\begin{aligned} \left\|\mathbf{K}_r^i\right\| &= \left\|\mathbf{u}_r^i \bar{\mathbf{v}}_r^{iT}\right\| \leqslant \left\|\mathbf{u}_r^i\right\| \cdot \left\|\bar{\mathbf{v}}_r^{iT}\right\| \sim \begin{cases} O(1) & \text{for } r>0, \\ O(\varepsilon^2) & \text{for } r<0, \end{cases} \\ \left\|\breve{\mathbf{K}}_r^i\right\| &= \left\|\mathbf{u}_r^i \bar{\breve{\mathbf{v}}}_r^{iT}\right\| \leqslant \left\|\mathbf{u}_r^i\right\| \cdot \left\|\bar{\breve{\mathbf{v}}}_r^{iT}\right\| \sim \begin{cases} O(\varepsilon) & \text{for } r>0, \\ O(\varepsilon) & \text{for } r<0, \end{cases} \end{aligned} \tag{18a}$$

leading to

$$\begin{aligned} \left\|\mathbf{H}_{gp}(\mathrm{j}\omega)\right\| &\sim \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{O(1)}{\left|\mathrm{j}\omega - \lambda_r^i\right|} + \frac{O(\varepsilon^2)}{\left|\mathrm{j}\omega - \lambda_{-r}^i\right|} \right], \\ \left\|\mathbf{H}_{\hat{g}p}(\mathrm{j}\omega)\right\| &\sim \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{O(\varepsilon)}{\left|\mathrm{j}\omega - \lambda_r^i\right|} + \frac{O(\varepsilon)}{\left|\mathrm{j}\omega - \lambda_{-r}^i\right|} \right]. \end{aligned} \tag{18b}$$

It can be concluded from Eq. (18b) that

1. ratio of the residue value of the normal dFRF between the positive and negative subscripted modes becomes $O(\varepsilon^2)$ The positive and negative subscripted modes are referred to as the strong

and weak modes, respectively. The strong modes, which are associated with the mean property of the rotor, are easily detected, but the weak modes, which are associated with the deviatoric property of the rotor, are hardly detected in the normal dFRF unless the degree of asymmetry becomes prominent.

2. The residue value of the r-dFRF for all, strong and weak, modes becomes the order of ε in magnitude. And the r-dFRF tends to vanish, as the asymmetry of the rotor decreases. Thus the magnitude of the r-dFRF, relative to that of the n-dFRF for strong modes, is a good indicator for presence of asymmetry in the rotor.

Note here that $O(1)$ may be smaller or larger in magnitude than $O(\varepsilon, \delta)$, depending upon the frequency ω of interest. However, $O(1)$ is independent of ε, δ, unlike $O(\varepsilon, \delta)$.

2.4. Relation between the modulated and rotating coordinates

The equation of motion of asymmetric rotor system in the rotating coordinates can be written as

$$\mathbf{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{C}\dot{\boldsymbol{q}}(t) + \boldsymbol{K}\boldsymbol{q}(t) = \boldsymbol{f}(t), \tag{19}$$

where

$$\boldsymbol{C} = \begin{bmatrix} \mathbf{C}_f + \mathrm{j}2\Omega\mathbf{M}_f & \mathbf{C}_r - \mathrm{j}2\Omega\mathbf{M}_r \\ \bar{\mathbf{C}}_r + \mathrm{j}2\Omega\bar{\mathbf{M}}_r & \bar{\mathbf{C}}_f - \mathrm{j}2\Omega\bar{\mathbf{M}}_f \end{bmatrix}_{2N\times 2N},$$

$$\boldsymbol{K} = \begin{bmatrix} \mathbf{K}_f + \mathrm{j}\Omega\mathbf{C}_f - \Omega^2\mathbf{M}_f & \mathbf{K}_r - \mathrm{j}\Omega\mathbf{C}_r - \Omega^2\mathbf{M}_r \\ \bar{\mathbf{K}}_r + \mathrm{j}\Omega\bar{\mathbf{C}}_r - \Omega^2\bar{\mathbf{M}}_r & \bar{\mathbf{K}}_f - \mathrm{j}\Omega\bar{\mathbf{C}}_f - \Omega^2\bar{\mathbf{M}}_f \end{bmatrix}_{2N\times 2N},$$

$$\boldsymbol{q}(t) = \begin{Bmatrix} \boldsymbol{p}(t) \\ \bar{\boldsymbol{p}}(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}(t)\mathrm{e}^{-\mathrm{j}\Omega t} \\ \bar{\mathbf{p}}(t)\mathrm{e}^{\mathrm{j}\Omega t} \end{Bmatrix}, \quad \boldsymbol{f}(t) = \begin{Bmatrix} \boldsymbol{g}(t) \\ \bar{\boldsymbol{g}}(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{g}(t)\mathrm{e}^{-\mathrm{j}\Omega t} \\ \bar{\mathbf{g}}(t)\mathrm{e}^{\mathrm{j}\Omega t} \end{Bmatrix}.$$

Here, the italicized, bold-faced letters represent the system matrices defined in the rotating coordinates. The relation between the stationary coordinates $\mathbf{p}(t)$ and the rotating coordinates $\boldsymbol{p}(t)$ is defined as $\mathbf{p}(t) = \boldsymbol{p}(t)\mathrm{e}^{\mathrm{j}\Omega t}$. Thus, it follows that $\mathbf{g}(t) = \boldsymbol{g}(t)\mathrm{e}^{\mathrm{j}\Omega t}$. And it can be easily shown that $|\mathbf{D}(\lambda)| = |\boldsymbol{D}(\lambda - \mathrm{j}\Omega)| = 0$, where $\boldsymbol{D}(\lambda) = \lambda^2\mathbf{M} + \lambda\boldsymbol{C} + \boldsymbol{K}$. It implies that the eigenvalues in the rotating coordinates are shifted by $(-\mathrm{j}\Omega)$ from those in the modulated stationary coordinates. Thus, if the eigenvalues associated with the modulated stationary and rotating complex coordinates are defined as λ and μ, respectively, they satisfy the relation, $\mu = \lambda - \mathrm{j}\Omega$. The relations between the eigenvalues for both coordinate systems are summarized in Table 1.

Table 1
Eigenvalue relations

	Modulated stationary coordinates	Rotating coordinates
Eigenvalue	λ_r^i	$\mu_r^i = \lambda_r^i - \mathrm{j}\Omega$
Paired value	$\lambda_{-r}^i = \bar{\lambda}_r^i + \mathrm{j}2\Omega$	$\mu_{-r}^i = \bar{\mu}_r^i$

3. Modal analysis of simple asymmetric rotor system

This section is provided to analytically illustrate the modal analysis procedure using modulated coordinates, which has been rigorously developed in the previous section, with a simple, yet comprehensive, asymmetric rotor system. The proposed method is also compared with the method based on formulation in the rotating coordinates. The equation of motion for the system shown in Fig. 1 can be written, in the stationary coordinate system, as

$$\ddot{p}(t)+\varepsilon e^{j2\Omega t}\ddot{\bar{p}}(t)-j\alpha\Omega\dot{p}(t)+j2\varepsilon\Omega e^{j2\Omega t}\dot{\bar{p}}(t)+\omega_0^2 p(t)+\delta\omega_0^2 e^{j(2\Omega t+2\varphi)}\bar{p}(t)=g(t)/J_t \tag{20}$$

where $p(t)$ and $g(t)$ are the complex (angular displacement) coordinate and the corresponding force (torque) defined in the stationary coordinate system; $\varepsilon=\left(J_\xi-J_\eta\right)/(2J_t)$ and $\delta=\left(k_\xi-k_\eta\right)/(2k)$ indicate the degrees of the inertia and stiffness asymmetry, respectively, $J_t=\left(J_\xi+J_\eta\right)/2$ and $k=\left(k_\xi+k_\eta\right)/2$ are the mean inertia and stiffness, respectively, J_ξ and J_η are the two principal diametrical mass moments of inertia of the disk, J_p is the polar mass moment of inertia of the disk, k_ξ and k_η are the two principal stiffness coefficients, $\alpha=J_p/J_t$ is the disk shape factor. The value of α lies between 0 (for an infinitely long disk) and 2 (for an infinitesimally thin disk). The undamped natural frequency $\omega_0=\sqrt{k/J_t}$ and φ is the angle between the principal axes of the shaft area moment and of the disk mass moment of inertia. The equation of motion (20) can be rewritten, using the modulated complex stationary coordinates, as

$$\begin{bmatrix}1 & \varepsilon\\ \varepsilon & 1\end{bmatrix}\begin{Bmatrix}\ddot{p}(t)\\ \ddot{\tilde{p}}(t)\end{Bmatrix}+\begin{bmatrix}-j\alpha\Omega & -j2\varepsilon\Omega\\ -j2\varepsilon\Omega & -j(4-\alpha)\Omega\end{bmatrix}\begin{Bmatrix}\dot{p}(t)\\ \dot{\tilde{p}}(t)\end{Bmatrix}$$
$$+\begin{bmatrix}\omega_0^2 & \delta e^{2j\varphi}\omega_0^2\\ \delta e^{-2j\varphi}\omega_0^2 & \omega_0^2-2(2-\alpha)\Omega^2\end{bmatrix}\begin{Bmatrix}p(t)\\ \tilde{p}(t)\end{Bmatrix}=\begin{Bmatrix}g(t)\\ \tilde{g}(t)\end{Bmatrix}. \tag{21}$$

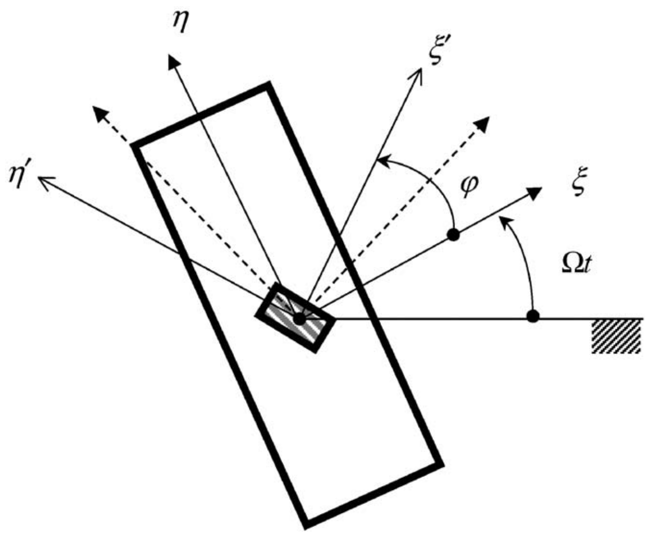


Fig. 1. A simple rotor with asymmetric stiffness and inertia [4].

The characteristic equation associated with Eq. (21) becomes

$$\left|\mathbf{D}(\lambda)\right| = \left(\lambda - \lambda_1^F\right)\left(\lambda - \lambda_{-1}^F\right)\left(\lambda - \lambda_1^B\right)\left(\lambda - \lambda_{-1}^B\right) = 0, \tag{22a}$$

from which the eigenvalues (latent roots) are obtained as

$$\lambda_{\pm 1}^B = \pm \mathrm{j}\omega_0\sqrt{\frac{b_2 - \sqrt{b_2^2 - b_1 b_3}}{b_1}} + \mathrm{j}\Omega, \quad \lambda_{\pm 1}^F = \pm \mathrm{j}\omega_0\sqrt{\frac{b_2 + \sqrt{b_2^2 - b_1 b_3}}{b_1}} + \mathrm{j}\Omega, \tag{22b}$$

where

$$b_1 = 1 - \varepsilon^2, \quad b_2 = 1 - \varepsilon\delta\cos 2\varphi + \left(\Omega/\omega_0\right)^2\left(1 - \alpha + \alpha^2/2 - \varepsilon^2\right),$$
$$b_3 = 1 - \delta^2 - 2\left(\Omega/\omega_0\right)^2\left\{1 - \alpha + \varepsilon\delta\cos 2\varphi\right\} + \left(\Omega/\omega_0\right)^4(1 - \alpha + \varepsilon)(1 - \alpha - \varepsilon).$$

It can be easily proven, from Eq. (22), that $\lambda_{-1}^B = \bar{\lambda}_1^B + \mathrm{j}2\Omega$ and $\lambda_{-1}^F = \bar{\lambda}_1^F + \mathrm{j}2\Omega$. The unstable threshold speeds Ω_t can be obtained, from the stability limit condition $b_3 = 0$ that holds for small ε and δ such that $b_1 > 0$ and $b_2 > 0$, as

$$\left(\frac{\Omega_{t1,t2}}{\omega_0}\right)^2 = \frac{(1 - \alpha + \varepsilon\delta\cos 2\varphi) \pm \sqrt{(1 - \alpha + \varepsilon\delta\cos 2\varphi)^2 - (1 - \alpha + \varepsilon)(1 - \alpha - \varepsilon)\left(1 - \delta^2\right)}}{(1 - \alpha + \varepsilon)(1 - \alpha - \varepsilon)}. \tag{23}$$

Using Eq. (4), we obtain

$$\frac{u_r^i}{\bar{u}_{-r}^i} = -\frac{d_r^i}{c_r^i} = -\frac{b_r^i}{a_r^i}, \quad \frac{\bar{v}_r^i}{v_{-r}^i} = -\frac{d_r^i}{b_r^i} = -\frac{c_r^i}{a_r^i}, \tag{24}$$

where, for $r = \pm 1$, $i = B, F$,

$$\mathbf{u}_{cr}^i = \begin{Bmatrix} u_r^i \\ \breve{u}_r^i \end{Bmatrix} = \begin{Bmatrix} u_r^i \\ \bar{u}_{-r}^i \end{Bmatrix}, \quad \bar{\mathbf{v}}_{cr}^i = \begin{Bmatrix} \bar{v}_r^i \\ \bar{\breve{v}}_r^i \end{Bmatrix} = \begin{Bmatrix} \bar{v}_r^i \\ v_{-r}^i \end{Bmatrix},$$

$$a_r^i = \left(\frac{\lambda_r^i}{\omega_0}\right)^2 - \mathrm{j}\alpha\left(\frac{\Omega}{\omega_0}\right)\left(\frac{\lambda_r^i}{\omega_0}\right) + 1 = \frac{\left(\lambda_r^i - \lambda_{r0}^F\right)\left(\lambda_r^i - \lambda_{r0}^B\right)}{\omega_0^2},$$

$$b_r^i = \varepsilon\left\{\left(\frac{\lambda_r^i}{\omega_0}\right) - \mathrm{j}2\left(\frac{\Omega}{\omega_0}\right)\right\}\left(\frac{\lambda_r^i}{\omega_0}\right) + \delta e^{\mathrm{j}2\varphi},$$

$$c_r^i = \varepsilon\left\{\left(\frac{\lambda_r^i}{\omega_0}\right) - \mathrm{j}2\left(\frac{\Omega}{\omega_0}\right)\right\}\left(\frac{\lambda_r^i}{\omega_0}\right) + \delta e^{-\mathrm{j}2\varphi},$$

$$d_r^i = \left(\frac{\lambda_r^i}{\omega_0}\right)^2 - \mathrm{j}(4 - \alpha)\left(\frac{\Omega}{\omega_0}\right)\left(\frac{\lambda_r^i}{\omega_0}\right) - 2(2 - \alpha)\left(\frac{\Omega}{\omega_0}\right)^2 + 1$$
$$= \frac{\left(\lambda_r^i - \bar{\lambda}_{r0}^F - \mathrm{j}2\Omega\right)\left(\lambda_r^i - \bar{\lambda}_{r0}^B - \mathrm{j}2\Omega\right)}{\omega_0^2} = \frac{\left(\lambda_r^i - \lambda_{r0}^F\right)\left(\lambda_r^i - \lambda_{r0}^B\right)}{\omega_0^2}.$$

Here, λ_{r0}^F and λ_{r0}^B are the eigenvalues associated with the forward and backward modes, respectively, of the isotropic rotor whose equation of motion is given, letting $\delta = \varepsilon = 0$ in Eq. (20), as

$$\ddot{\mathrm{p}}(t) - \mathrm{j}\alpha\Omega\dot{p}(t) + \omega_0^2 p(t) = g(t)/J_t. \tag{25}$$

Using the result in Eq. (6), we can derive

$$K_r^i = \mathbf{u}_r^i \bar{\mathbf{v}}_r^i = \frac{d_r^i}{\Lambda_r^i}, \quad \breve{K}_r^i = \mathbf{u}_r^i \mathbf{v}_{-r}^i = -\frac{b_r^i}{\Lambda_r^i}, \tag{26}$$

where

$$\Lambda_r^i = 2\lambda_r^i\left(a_r^i + d_r^i\right) + \mathrm{j}(2-\alpha)\left(d_r^i - a_r^i\right)\Omega - 2\varepsilon\lambda_r^i\left(b_r^i + c_r^i\right).$$

Here, K_r^i and $\breve{K}_r^i$ are the residues associated with the mode λ_r^i. Note that, for the special cases of $\varphi = 0$ and $\varphi = \pi/2$, i.e. when the principal axes of the diametrical moment of inertia and the angular stiffness are aligned, it holds $u_r^i/\bar{u}_{-r}^i = \bar{v}_r^i/v_{-r}^i$, because the system matrices become symmetric, which does not hold in general. Using the above relation (26), we can express the dFRFs as

$$\begin{aligned}
H_{gp}(\mathrm{j}\omega) &= \frac{u_1^F\bar{v}_1^F}{\mathrm{j}\omega - \lambda_1^F} + \frac{u_{-1}^F\bar{v}_{-1}^F}{\mathrm{j}\omega - \lambda_{-1}^F} + \frac{u_1^B\bar{v}_1^B}{\mathrm{j}\omega - \lambda_1^B} + \frac{u_{-1}^B\bar{v}_{-1}^B}{\mathrm{j}\omega - \lambda_{-1}^B} \\
&= \frac{K_1^F}{\mathrm{j}\omega - \lambda_1^F} + \frac{K_{-1}^F}{\mathrm{j}\omega - \lambda_{-1}^F} + \frac{K_1^B}{\mathrm{j}\omega - \lambda_1^B} + \frac{K_{-1}^B}{\mathrm{j}\omega - \lambda_{-1}^B}, \\
H_{\tilde{g}p}(\mathrm{j}\omega) &= \frac{u_1^F\bar{\tilde{v}}_1^F}{\mathrm{j}\omega - \lambda_1^F} + \frac{u_{-1}^F\bar{\tilde{v}}_{-1}^F}{\mathrm{j}\omega - \lambda_{-1}^F} + \frac{u_1^B\bar{\tilde{v}}_1^B}{\mathrm{j}\omega - \lambda_1^B} + \frac{u_{-1}^B\bar{\tilde{v}}_{-1}^B}{\mathrm{j}\omega - \lambda_{-1}^B} \\
&= \frac{\breve{K}_1^F}{\mathrm{j}\omega - \lambda_1^F} + \frac{\breve{K}_{-1}^F}{\mathrm{j}\omega - \lambda_{-1}^F} + \frac{\breve{K}_1^B}{\mathrm{j}\omega - \lambda_1^B} + \frac{\breve{K}_{-1}^B}{\mathrm{j}\omega - \lambda_{-1}^B}.
\end{aligned} \tag{27}$$

On the other hand, transformation of Eq. (21) into the rotating coordinates leads to

$$\begin{aligned}
&\begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix} \begin{Bmatrix} \ddot{p}(t) \\ \ddot{\tilde{p}}(t) \end{Bmatrix} + \begin{bmatrix} \mathrm{j}(2-\alpha)\Omega & 0 \\ 0 & -\mathrm{j}(2-\alpha)\Omega \end{bmatrix} \begin{Bmatrix} \dot{p}(t) \\ \dot{\tilde{p}}(t) \end{Bmatrix} \\
&+ \begin{bmatrix} \omega_0^2 + (\alpha-1)\Omega^2 & \delta\mathrm{e}^{\mathrm{j}2\varphi}\omega_0^2 + \varepsilon\Omega^2 \\ \delta\mathrm{e}^{-\mathrm{j}2\varphi}\omega_0^2 + \varepsilon\Omega^2 & \omega_0^2 + (\alpha-1)\Omega^2 \end{bmatrix} \begin{Bmatrix} p(t) \\ \tilde{p}(t) \end{Bmatrix} = \begin{Bmatrix} g(t) \\ \tilde{g}(t) \end{Bmatrix}.
\end{aligned} \tag{28}$$

The characteristic equation associated with Eq. (28) becomes

$$\left|\mathbf{D}(\lambda)\right| = \left(\mu - \mu_1^F\right)\left(\mu - \mu_{-1}^F\right)\left(\mu - \mu_1^B\right)\left(\mu - \mu_{-1}^B\right) = 0, \tag{29a}$$

from which the eigenvalues are obtained as

$$\mu_{\pm 1}^B = \pm\mathrm{j}\omega_0\sqrt{\frac{b_2 - \sqrt{b_2^2 - b_1 b_3}}{b_1}}, \quad \mu_{\pm 1}^F = \pm\mathrm{j}\omega_0\sqrt{\frac{b_2 + \sqrt{b_2^2 - b_1 b_3}}{b_1}}. \tag{29b}$$

It can be easily proven, from Eqs. (22) and (29), that $\mu_{\pm 1}^B = \bar{\mu}_{\mp 1}^B, \mu_{\pm 1}^F = \bar{\mu}_{\mp 1}^F$ and $\mu_{\pm 1}^B = \lambda_{\pm 1}^B - \mathrm{j}\Omega, \mu_{\pm 1}^F = \lambda_{\pm 1}^F - \mathrm{j}\Omega$. On the other hand, the bi-orthonormalized latent vectors turn out to be the same as those obtained in the modulated coordinates, i.e.

$$\boldsymbol{u}^{B,F}_{c\pm 1}=\begin{Bmatrix} u^{B,F}_{\pm 1}\\ \bar{u}^{B,F}_{\mp 1}\end{Bmatrix}=\mathbf{u}^{B,F}_{c\pm 1}=\begin{Bmatrix} u^{B,F}_{\pm 1}\\ \bar{u}^{B,F}_{\mp 1}\end{Bmatrix},\quad \bar{\boldsymbol{v}}^{B,F}_{c\pm 1}=\begin{Bmatrix} \bar{v}^{B,F}_{\pm 1}\\ v^{B,F}_{\mp 1}\end{Bmatrix}=\bar{\mathbf{v}}^{B,F}_{c\pm 1}=\begin{Bmatrix} \bar{v}^{B,F}_{\pm 1}\\ v^{B,F}_{\mp 1}\end{Bmatrix}. \tag{30}$$

Then the dFRFs in the rotating coordinates are given as

$$\begin{aligned}
H_{gp}(\mathrm{j}\omega) &= \frac{u_1^F\bar{v}_1^F}{\mathrm{j}\omega-\mu_1^F}+\frac{u_{-1}^F\bar{v}_{-1}^F}{\mathrm{j}\omega-\mu_{-1}^F}+\frac{u_1^B\bar{v}_1^B}{\mathrm{j}\omega-\mu_1^B}+\frac{u_{-1}^B\bar{v}_{-1}^B}{\mathrm{j}\omega-\mu_{-1}^B}\\
&= \frac{K_1^F}{\mathrm{j}\omega-\mu_1^F}+\frac{K_{-1}^F}{\mathrm{j}\omega-\mu_{-1}^F}+\frac{K_1^B}{\mathrm{j}\omega-\mu_1^B}+\frac{K_{-1}^B}{\mathrm{j}\omega-\mu_{-1}^B},\\
H_{\hat{g}p}(j\omega) &= \frac{u_1^F\bar{\bar{v}}_1^F}{\mathrm{j}\omega-\mu_1^F}+\frac{u_{-1}^F\bar{\bar{v}}_{-1}^F}{\mathrm{j}\omega-\mu_{-1}^F}+\frac{u_1^B\bar{\bar{v}}_1^B}{\mathrm{j}\omega-\mu_1^B}+\frac{u_{-1}^B\bar{\bar{v}}_{-1}^B}{\mathrm{j}\omega-\mu_{-1}^B}\\
&= \frac{\check{K}_1^F}{\mathrm{j}\omega-\mu_1^F}+\frac{\check{K}_{-1}^F}{\mathrm{j}\omega-\mu_{-1}^F}+\frac{\check{K}_1^B}{\mathrm{j}\omega-\mu_1^B}+\frac{\check{K}_{-1}^B}{\mathrm{j}\omega-\mu_{-1}^B}.
\end{aligned} \tag{31}$$

Here, $H_{gp}(\mathrm{j}\omega)$ and $H_{\hat{g}p}(\mathrm{j}\omega)$ represent the normal and reverse dFRFs defined in the rotating coordinates, respectively. Thus, the relation between dFRFs defined from the modulated and rotating coordinates is given by

$$\begin{aligned}
H_{gp}(\mathrm{j}(\omega+\Omega)) &= \sum_{\substack{r=\pm 1\\ i=B,F}}\frac{u_r^i\bar{v}_r^i}{\mathrm{j}\omega+\mathrm{j}\Omega-\lambda_r^i}=H_{gp}(\mathrm{j}\omega)=\sum_{\substack{r=\pm 1\\ i=B,F}}\frac{u_r^i\bar{v}_r^i}{\mathrm{j}\omega-\mu_r^i},\\
H_{\bar{g}p}(\mathrm{j}(\omega+\Omega)) &= \sum_{\substack{r=\pm 1\\ i=B,F}}\frac{u_r^i\bar{\bar{v}}_r^i}{\mathrm{j}\omega+\mathrm{j}\Omega-\lambda_r^i}=H_{\hat{g}p}(\mathrm{j}\omega)=\sum_{\substack{r=\pm 1\\ i=B,F}}\frac{u_r^i\bar{\bar{v}}_r^i}{\mathrm{j}\omega-\mu_r^i}.
\end{aligned} \tag{32}$$

In summary, Eq. (27) and, equivalently, Eq. (31) can be reduced to the form of

$$\begin{aligned}
\left|H_{gp}(\mathrm{j}\omega)\right| &\sim \frac{O(1)}{\left|\mathrm{j}\omega-\lambda_1^F\right|}+\frac{O(\varepsilon,\delta)^2}{\left|\mathrm{j}\omega-\lambda_{-1}^F\right|}+\frac{O(1)}{\left|\mathrm{j}\omega-\lambda_1^B\right|}+\frac{O(\varepsilon,\delta)^2}{\left|\mathrm{j}\omega-\lambda_{-1}^B\right|}\\
\left|H_{\bar{g}p}(\mathrm{j}\omega)\right| &\sim \frac{O(\varepsilon,\delta)}{\left|\mathrm{j}\omega-\lambda_1^F\right|}+\frac{O(\varepsilon,\delta)}{\left|\mathrm{j}\omega-\lambda_{-1}^F\right|}+\frac{O(\varepsilon,\delta)}{\left|\mathrm{j}\omega-\lambda_1^B\right|}+\frac{O(\varepsilon,\delta)}{\left|\mathrm{j}\omega-\lambda_{-1}^B\right|}.
\end{aligned} \tag{33}$$

From the above expressions (27) and (33) for the normal and reverse dFRFs, we can conclude that:

1. Eq. (22), we can easily verify the relation $\lambda_r^i-\lambda_{r0}^i\sim O(\varepsilon,\delta)^2$. It implies that the perturbation of the eigenvalue due to the presence of asymmetry (ε,δ) in the rotor model is the order of $(\varepsilon,\delta)^2$.
2. It holds

$$\frac{K_{-1}^i}{K_1^i}=\left(\frac{d_{-1}^i}{d_1^i}\right)\frac{\Lambda_1^i}{\Lambda_{-1}^i}\cong\frac{\left(\lambda_{-1}^i-\lambda_{-1}^{oF}\right)\left(\lambda_{-1}^i-\lambda_{-1}^{oB}\right)}{\left(\lambda_1^i-\lambda_{-1}^{oF}\right)\left(\lambda_1^i-\lambda_{-1}^{oB}\right)}\sim\frac{O(\varepsilon,\delta)^2}{O(1)}\sim O(\varepsilon,\delta)^2,$$

i.e. the ratio of the residue value of the normal dFRF between the strong and weak modes becomes $O(\varepsilon,\delta)^2$.

3. Since $\check{K}_r^i = -b_r^i/\Lambda_r^i \sim O(\varepsilon, \delta)$, the residue value of the r-dFRF for all, strong and weak, modes becomes the order of (ε, δ) in magnitude.

In order to demonstrate the analytical findings with the simple asymmetric rotor system, the parameter values given in Table 2 are used in the simulation, ignoring the inertia asymmetry (ε). Figs. 2(a) and (b), the whirl charts, are the plots of the imaginary part of the eigenvalues obtained, respectively, in the modulated stationary and the rotating coordinates. In the plots, the strong (marked $\lambda_1^B, \lambda_1^F, \mu_1^B, \mu_1^F$) and weak (marked $\lambda_{-1}^B, \lambda_{-1}^F, \mu_{-1}^B, \mu_{-1}^F$) modes are represented by thick and thin solid lines, respectively. Note that the whirl charts in the modulated stationary and the rotating coordinates are symmetric with respect to $\mathrm{Im}(\lambda) = \Omega$ (synchronous to the rotational speed) and $\mathrm{Im}(\mu) = 0$, respectively, since it holds $\lambda_{-1}^B - \mathrm{j}\Omega = \overline{(\lambda_1^B - \mathrm{j}\Omega)}$, $\lambda_{-1}^F - \mathrm{j}\Omega = \overline{(\lambda_1^F - \mathrm{j}\Omega)}$ and $\mu_{-1}^B = \bar{\mu}_1^B, \mu_{-1}^F = \bar{\mu}_1^F$. Comparison of Figs. 2(a) and (b) also confirms the relations $\mu_{\pm 1}^B = \lambda_{\pm 1}^B - \mathrm{j}\Omega, \mu_{\pm 1}^F = \lambda_{\pm 1}^F - \mathrm{j}\Omega$.

Figs. 3 and 4 show the cascade plots for the dFRFs of the simple asymmetric rotor in the modulated stationary and the rotating coordinates, respectively. Note that the dFRFs in the rotating coordinates can be obtained by shifting the corresponding dFRFs in the modulated

Table 2
Numerical data for the analysis model

Parameter	α	ε	φ	ω_0	δ	Ω
Data	0.6	0	0	1.0 (rad/s)	0.5	0–5 (rad/s)

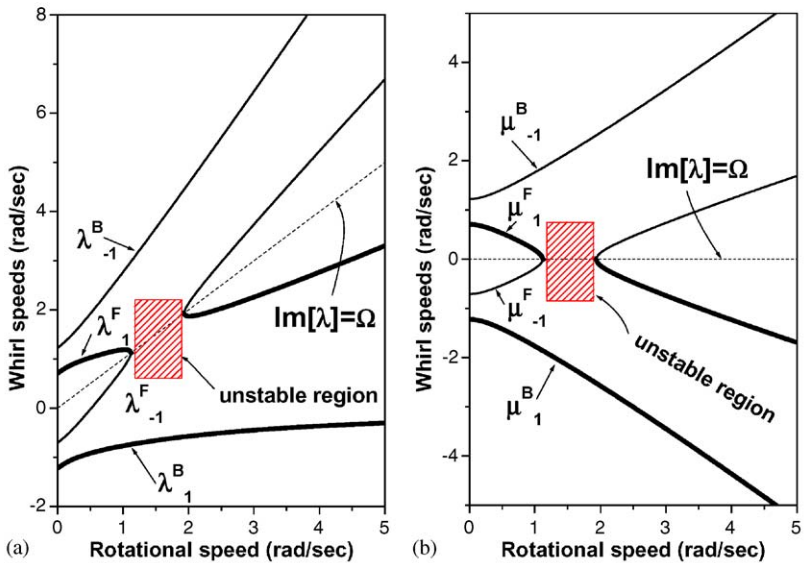


Fig. 2. Whirl charts of the simple asymmetric rotor system in the (a) modulated and (b) rotating coordinates.

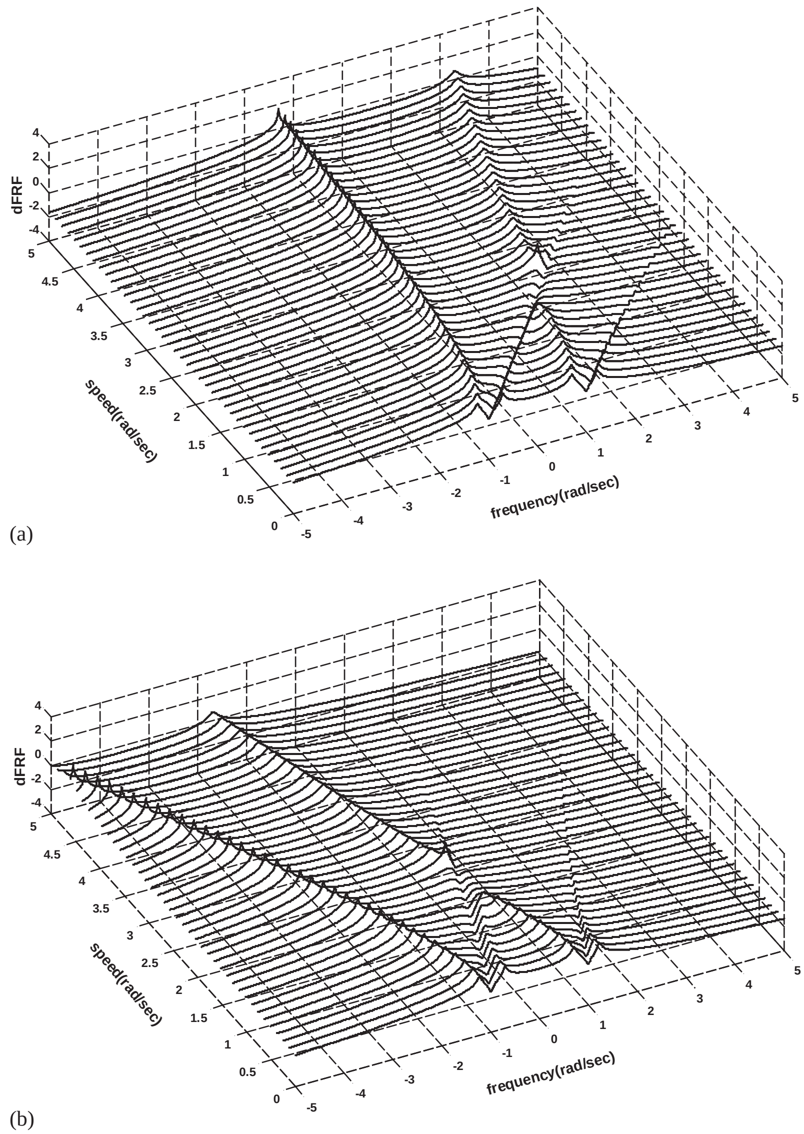
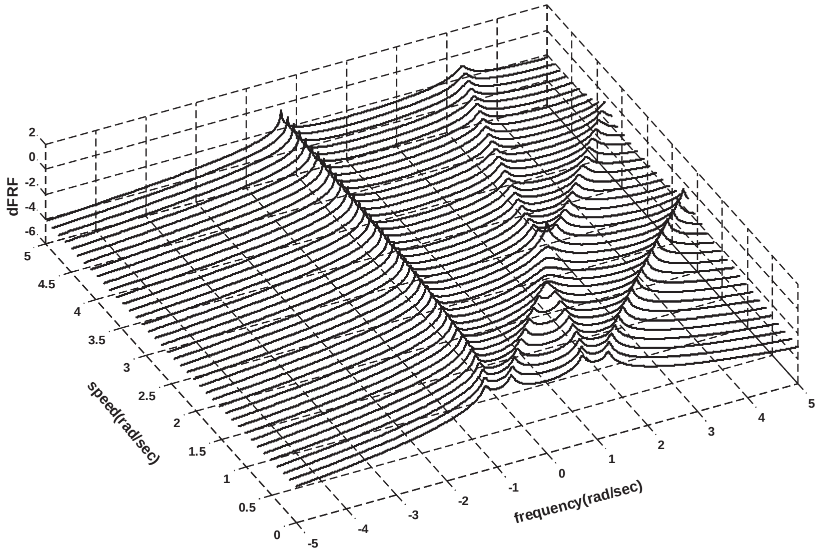


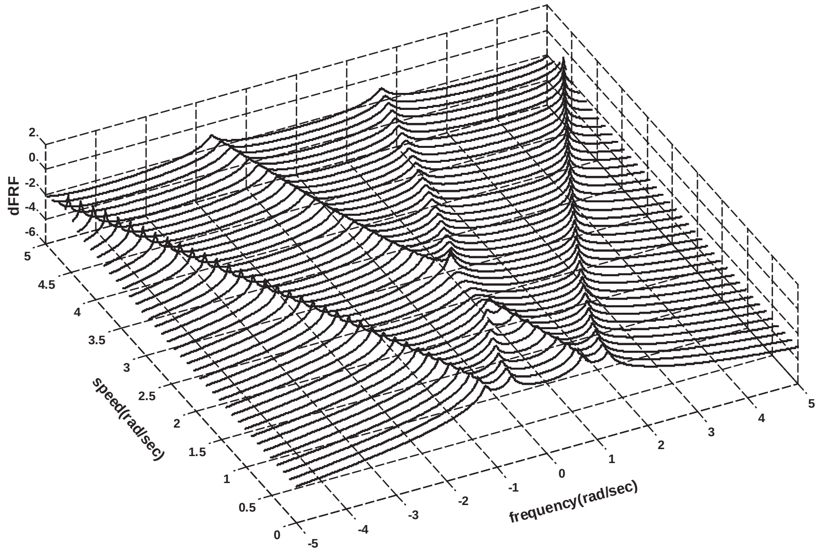
Fig. 3. Cascade plots for n-dFRFs of simple asymmetric rotor with isotropic stator in the (a) modulated and (b) rotating coordinates (modal damping ratio of 0.03 was imposed solely for plotting convenience).

stationary coordinates by $-\Omega$. In the plots of n-dFRFs, only the two strong (positive subscripted) modes $(\lambda_1^B, \lambda_1^F)$ are prominent, while the two weak (negative subscripted) modes $(\lambda_{-1}^B, \lambda_{-1}^F)$ are hardly observed. On the other hand, the weak modes are as significant as the strong modes in the r-dFRFs.

In summary, this section analytically and numerically shows the modal analysis procedure of the proposed method and compares the proposed method and the conventional method based on the rotating coordinate transformation, confirming that the proposed method provides the same



(a)



(b)

Fig. 4. Cascade plots for shifted r-dFRFs of simple asymmetric rotor with isotropic stator in the (a) modulated and (b) rotating coordinates (modal damping ratio of 0.03 was imposed solely for plotting convenience).

useful information as the conventional method based on formulation in the rotating coordinates. Note that transformation of the equation of motion for asymmetric rotor system defined in the stationary coordinates into that defined in the rotating coordinates is also a kind of modulation technique, which has been proven useful but confined only to asymmetric rotor system with isotropic stator. However, the proposed method can be extended, without loss of generality, to general rotor systems whose rotating and stationary parts both possess asymmetric properties.

For general rotors, transformation of the periodically time-varying equation of motion to the rotating coordinates does not lead to a time-invariant equation. Thus, the conventional modal analysis method based on the time-invariant equation of motion cannot be directly applied to general rotors. The proposed method becomes still effective for general rotors, which will be extensively treated in a separate paper.

4. Numerical example: a flexible asymmetric rotor

To demonstrate the applicability of the proposed modal analysis method developed for asymmetric rotors, a numerical example is treated with a flexible asymmetric rotor, having an open transverse crack. The finite element model of the rotor is shown in Fig. 5, and the material and geometrical properties are listed in Table 3. The model consists of 26 Rayleigh beam elements, two rigid disks, and two isotropic bearing supports. We assume there exists an open transverse crack in the shaft, which causes bending stiffness asymmetry in the shaft [10]. Here, an open crack with the crack depth to shaft diameter ratio $a/D = 0.48$ is assumed to develop at node #12. It is well known in the literature on fracture that the plastic deformation near the crack edge or the oxide layer formed on the cracked region resists the crack from being fully closed back again [11]. Thus, an open crack model may be often valid. Furthermore, for the purpose of effective diagnosis of cracks, an electromagnetic exciter can be employed to properly generate a harmonic excitation synchronous to the shaft rotational frequency so that the synchronous excitation allows the breathing crack completely open during the shaft revolution. In this case, the rotor behaves like a typical asymmetric rotor, whose dynamics can be easily analyzed by the proposed method [12].

In this example, the bending stiffness reduction due to stress concentration around the crack is taken into account using the stress intensity factor [13,14]. The whirl charts are plotted in Figs. 6(a) and (b), respectively, for the modulated complex stationary coordinates and the complex rotating coordinates over th rotational speed range of 0–10,000 rpm. Comparison of the two results confirms the relation, $\mu_r^i = \lambda_r^i - \mathrm{j}\Omega$. The hatched regions marked in the figures indicate the unstable speed regions where the real parts of the eigenvalues become positive. The two figures provide identical information regarding the stability issue.

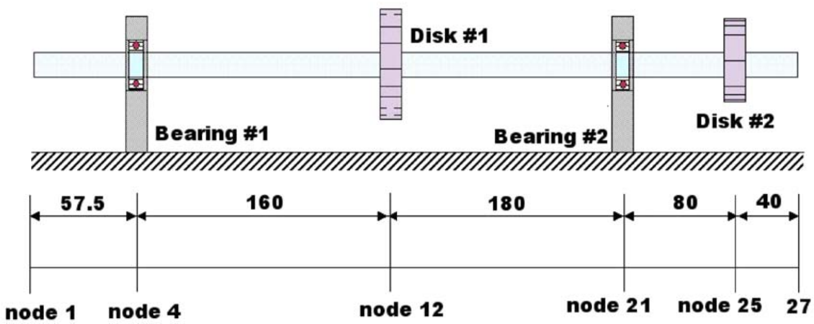


Fig. 5. Flexible rotor configuration and geometry.

Table 3
Specifications of the numerical mode

Mesh data			
# of elements	= 26		
# of disks	= 2		
# of bearings	= 2		
Shaft			
Length = 51 cm, Diameter = 1.2 cm			
Density = 7806 kg/m^3			
Young's modulus = 2.08×10^{11} N/m^2			
Location (m)	Mass (kg)	Polar moment of inertia (kg m^2)	Dia. moment of inertia (kg m^2)
Disk			
0.21	1.236	1.2×10^{-3}	6.8×10^{-4}
0.476	0.857	0.9×10^{-3}	3.5×10^{-4}
Node number	Stiffness (N/m)	Damping (N s/m)	
Bearings			
4.21	$k_{yy} = k_{zz} = 5 \times 10^8$	$c_{yy} = c_{zz} = 4.5 \times 10^3$	
(identical)	$k_{yz} = k_{zy} = 0$	$c_{yy} = c_{zz} = 0$	

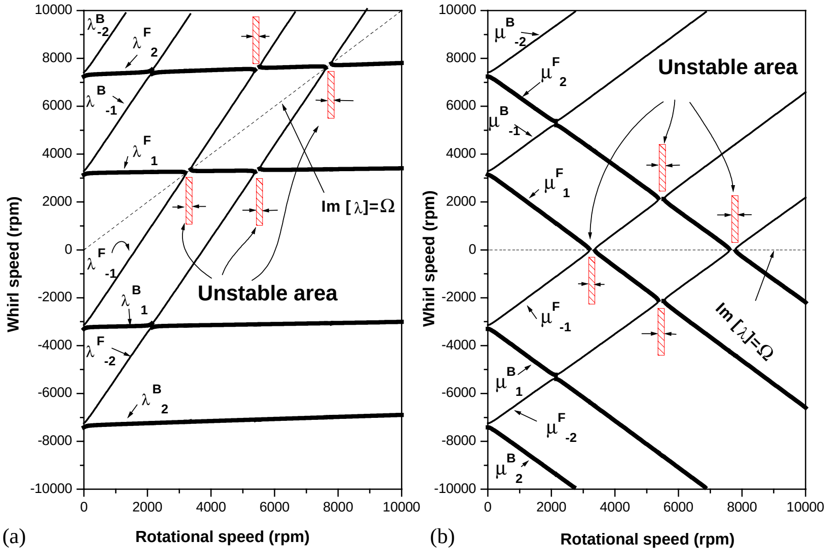


Fig. 6. Whirl charts of the open cracked flexible asymmetric rotor in the (a) modulated and (b) rotating coordinates: crack at node #12 with a/D = 0.48.

Fig. 7 shows the cascade plots of the n/r-dFRFs at node #12, when the excitation force is given at node #25, the location of disk #2. The operating rotational speed is assumed to be 600 rpm (10 Hz). In the n-dFRF plot, the four strong modes ($\lambda_r^i, i = B, F, r = 1, 2$) are dominant, whilst the weak modes are hardly observed. On the other hand, the r-dFRF plot, unlike the n-dFRF plot, equally well reveals all the eight modes.

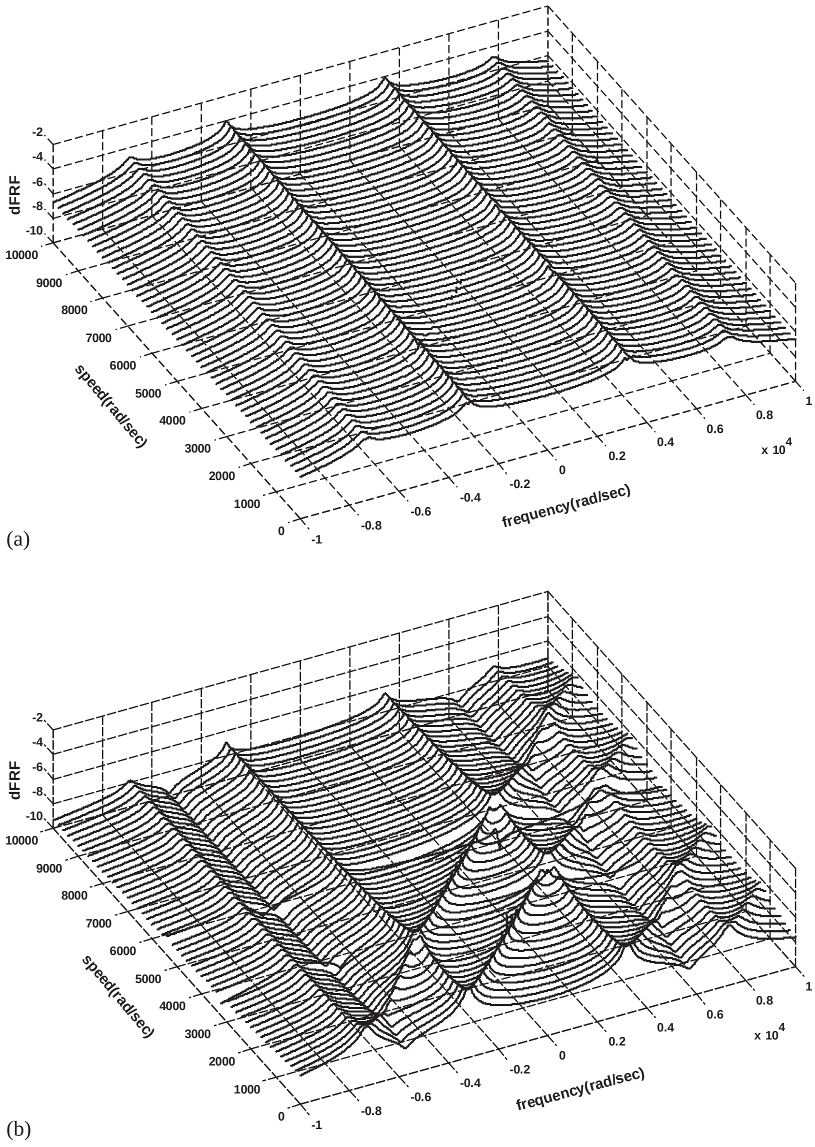


Fig. 7. Cascade plots of (a) normal and (b) shifted reverse dFRFs of the flexible rotor with a crack: number of used modes = 12; excitation node 25; sensor node 13 (modal damping ratio of 0.02 was imposed solely for plotting convenience).

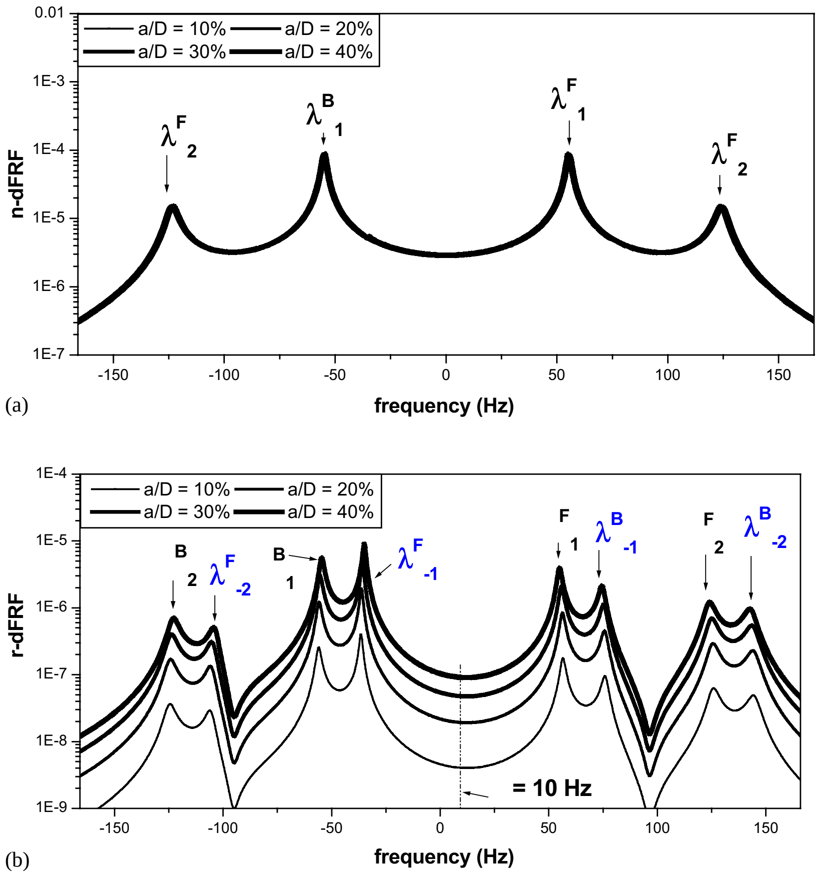


Fig. 8. (a) n-dFRFs and (b) shifted r-dFRFs of the cracked flexible rotor: sensor node = 13; exciter node = 25; crack location = 12: at 600 rpm.(modal damping ratio of 0.02 was imposed solely for plotting convenience).

Fig. 8 shows the dFRFs of the flexible rotor with the depth of crack varied. To better show the change of dFRFs due to the crack, modal damping ratio of 0.02 is imposed to every mode used in the calculation. The r-dFRFs are significantly changed due to the crack growth, while the normal dFRFs are little affected by the change of the crack depth. Figs. 7 and 8 imply that the r-dFRFs are very useful for identification of the presence and severity of crack in the shaft.

5. Concluding remarks

This paper presents a generalized modal analysis method for asymmetric rotor systems employing the modulated coordinates. An equivalent time-invariant equation of motion in the modulated complex stationary coordinates is derived. The proposed method provides a complete modal solution in the stationary coordinates for asymmetric rotor systems with isotropic stators. The characteristics of eigenvalues and latent vectors are theoretically investigated by using the equivalent time-invariant equation of motion.

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Appendix. Perturbation of latent roots

Let the perturbed latent roots and corresponding latent vectors be represented by

$$\lambda_r^i = \lambda_{r0}^i + \varepsilon\lambda_{r1}^i + \varepsilon^2\lambda_{r2}^i + O(\varepsilon^3), \quad r = \pm 1, \ldots \pm N, \tag{A.1}$$

$$\begin{Bmatrix} \mathbf{u}_r^i \\ \breve{\mathbf{u}}_r^i \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_{r0}^i + \varepsilon\mathbf{u}_{r1}^i + O(\varepsilon^2) \\ \breve{\mathbf{u}}_{r0}^i + \varepsilon\breve{\mathbf{u}}_{r1}^i + O(\varepsilon^2) \end{Bmatrix}, \tag{A.2}$$

$$\left\{ \bar{\mathbf{v}}_r^{i\mathrm{T}} \quad \bar{\breve{\mathbf{v}}}_r^{i\mathrm{T}} \right\} = \left\{ \bar{\mathbf{v}}_{r0}^{i\mathrm{T}} + \varepsilon\bar{\mathbf{v}}_{r1}^{i\mathrm{T}} + O(\varepsilon^2) \quad \bar{\breve{\mathbf{v}}}_{r0}^{i\mathrm{T}} + \varepsilon\bar{\breve{\mathbf{v}}}_{r1}^{i\mathrm{T}} + O(\varepsilon^2) \right\}. \tag{A.3}$$

Substituting Eqs. (A.1) and (A.2) into

$$\begin{bmatrix} \mathbf{D}_f(\lambda_r^i) & \varepsilon\tilde{\mathbf{D}}_r(\lambda_r^i) \\ \varepsilon\mathbf{D}_r(\lambda_r^i) & \tilde{\mathbf{D}}_f(\lambda_r^i) \end{bmatrix} \begin{Bmatrix} \mathbf{u}_r^i \\ \breve{\mathbf{u}}_r^i \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{A.4}$$

with

$$\begin{aligned}
\mathbf{D}_f(\lambda) &= \lambda^2\mathbf{M}_f + \lambda\mathbf{C}_f + \mathbf{K}_f, \\
\mathbf{D}_r(\lambda) &= \lambda^2\bar{\mathbf{M}}_r + \lambda\bar{\mathbf{C}}_r + \bar{\mathbf{K}}_r, \\
\tilde{\mathbf{D}}_r(\lambda) &= \lambda^2\mathbf{M}_r + \lambda(\mathbf{C}_r - \mathrm{j}4\Omega\mathbf{M}_r) + \mathbf{K}_r - \mathrm{j}2\Omega\mathbf{C}_r - 4\Omega^2\mathbf{M}_r, \\
\tilde{\mathbf{D}}_f(\lambda) &= \lambda^2\bar{\mathbf{M}}_f + \lambda(\bar{\mathbf{C}}_f - \mathrm{j}4\Omega\bar{\mathbf{M}}_f) + \bar{\mathbf{K}}_f - \mathrm{j}2\Omega\bar{\mathbf{C}}_f - 4\Omega^2\bar{\mathbf{M}}_f.
\end{aligned} \tag{A.5}$$

We obtain

$$\begin{aligned}
&\mathbf{D}_f(\lambda_r^i)\mathbf{u}_r^i \\
&= \left[\left\{\lambda_{r0}^i + \varepsilon\lambda_{r1}^i + \varepsilon^2\lambda_{r2}^i + O(\varepsilon^3)\right\}^2\mathbf{M}_f + \left\{\lambda_{r0}^i + \varepsilon\lambda_{r1}^i + \varepsilon^2\lambda_{r2}^i + O(\varepsilon^3)\right\}\mathbf{C}_f + \mathbf{K}_f\right]\left\{\mathbf{u}_{r0}^i + \varepsilon\mathbf{u}_{r1}^i + O(\varepsilon^2)\right\} \\
&= \left[\lambda_{r0}^{i2}\mathbf{M}_f + \lambda_{r0}^i\mathbf{C}_f + \mathbf{K}_f + \varepsilon\lambda_{r1}^i\left\{2\lambda_{r0}^i\mathbf{M}_f + \mathbf{C}_f\right\} + \varepsilon^2\left\{(\lambda_{r1}^{i2} + 2\lambda_{r0}^i\lambda_{r2}^i)\mathbf{M}_f + \lambda_{r2}^i\mathbf{C}_f\right\}\right]\left\{\mathbf{u}_{r0}^i + \varepsilon\mathbf{u}_{r1}^i + O(\varepsilon^2)\right\} \\
&= \left\{\lambda_{r0}^{i2}\mathbf{M}_f + \lambda_{r0}^i\mathbf{C}_f + \mathbf{K}_f\right\}\mathbf{u}_{r0}^i + \varepsilon\left[\left\{\lambda_{r0}^{i2}\mathbf{M}_f + \lambda_{r0}^i\mathbf{C}_f + \mathbf{K}_f\right\}\mathbf{u}_{r1}^i + \lambda_{r1}^i\left\{2\lambda_{r0}^i\mathbf{M}_f + \mathbf{C}_f\right\}\mathbf{u}_{r0}^i\right] + O(\varepsilon^2), \\
&= \mathbf{D}_f(\lambda_{r0}^i)\mathbf{u}_{r0}^i + \varepsilon\left[\mathbf{D}_f(\lambda_{r0}^i)\mathbf{u}_{r1}^i + \lambda_{r1}^i\mathbf{D}_f'(\lambda_{r0}^i)\mathbf{u}_{r0}^i\right] + O(\varepsilon^2) \\
&\varepsilon\tilde{\mathbf{D}}_r(\lambda_r^i)\breve{\mathbf{u}}_r^i = \varepsilon\left[\tilde{\mathbf{D}}_r(\lambda_{r0}^i)\breve{\mathbf{u}}_{r0}^i\right] + O(\varepsilon^2), \\
&\varepsilon\mathbf{D}_r(\lambda_r^i)\mathbf{u}_r^i = \varepsilon\left[\mathbf{D}_r(\lambda_{r0}^i)\mathbf{u}_{r0}^i\right] + O(\varepsilon^2), \\
&\tilde{\mathbf{D}}_f(\lambda_r^i)\breve{\mathbf{u}}_r^i = \tilde{\mathbf{D}}_f(\lambda_{r0}^i)\breve{\mathbf{u}}_{r0}^i + \varepsilon\left[\tilde{\mathbf{D}}_f(\lambda_{r0}^i)\breve{\mathbf{u}}_{r1}^i + \lambda_{r1}^i\mathbf{D}_f'(\lambda_{r0}^i)\breve{\mathbf{u}}_{r0}^i\right] + O(\varepsilon^2)
\end{aligned} \tag{A.6}$$

leading to

$$\mathbf{D}_f(\lambda_r^i)\mathbf{u}_r^i + \varepsilon\tilde{\mathbf{D}}_r(\lambda_r^i)\breve{\mathbf{u}}_r^i = \mathbf{D}_f(\lambda_{r0}^i)\mathbf{u}_{r0}^i + \varepsilon\left[\mathbf{D}_f(\lambda_{r0}^i)\mathbf{u}_{r1}^i + \tilde{\mathbf{D}}_r(\lambda_{r0}^i)\breve{\mathbf{u}}_{r0}^i + \lambda_{r1}^i\mathbf{D}_f'(\lambda_{r0}^i)\mathbf{u}_{r0}^i\right] + O(\varepsilon^2) = \mathbf{0},$$

$$\varepsilon\mathbf{D}_r(\lambda_r^i)\mathbf{u}_r^i + \tilde{\mathbf{D}}_f(\lambda_r^i)\breve{\mathbf{u}}_r^i = \tilde{\mathbf{D}}_f(\lambda_{r0}^i)\breve{\mathbf{u}}_{r0}^i + \varepsilon\left[\tilde{\mathbf{D}}_f(\lambda_{r0}^i)\breve{\mathbf{u}}_{r1}^i + \mathbf{D}_r(\lambda_{r0}^i)\mathbf{u}_{r0}^i + \lambda_{r1}^i\tilde{\mathbf{D}}_f'(\lambda_{r0}^i)\breve{\mathbf{u}}_{r0}^i\right] + O(\varepsilon^2) = \mathbf{0}. \quad \text{(A.7)}$$

For $r>0$, since $\mathbf{D}_f(\lambda_{r0}^i)\mathbf{u}_{r0}^i = \mathbf{0}$ and $\breve{\mathbf{u}}_{r0}^i = \mathbf{0}$, it holds, from Eqs. (A.7) and (A.3),

$$\lambda_{r1}^i \cong -\frac{\left\{\bar{\mathbf{v}}_{r0}^{iT} + \varepsilon\bar{\mathbf{v}}_{r1}^{iT}\right\}\mathbf{D}_f(\lambda_{r0}^i)\mathbf{u}_{r1}^i}{\left\{\bar{\mathbf{v}}_{r0}^{iT} + \varepsilon\bar{\mathbf{v}}_{r1}^{iT}\right\}\mathbf{D}_f'(\lambda_{r0}^i)\mathbf{u}_{r0}^i} \cong \frac{\varepsilon\bar{\mathbf{v}}_{r1}^{iT}\mathbf{D}_f(\lambda_{r0}^i)\mathbf{u}_{r1}^i}{\bar{\mathbf{v}}_{r0}^{iT}\mathbf{D}_f'(\lambda_{r0}^i)\mathbf{u}_{r0}^i} \sim 0, \quad \text{(A.8)}$$

$$\breve{\mathbf{u}}_{r1}^i = -\tilde{\mathbf{D}}_f^{-1}(\lambda_{r0}^i)\mathbf{D}_r(\lambda_{r0}^i)\mathbf{u}_{r0}^i. \quad \text{(A.9)}$$

And, similarly, for $r<0$, since $\tilde{\mathbf{D}}_f(\lambda_{r0}^i)\mathbf{u}_{r0}^i = \mathbf{0}$ and $\mathbf{u}_{r0}^i = \mathbf{0}$, it holds, from Eqs. (A.7) and (A.3),

$$\lambda_{r1}^i \cong -\frac{\left\{\bar{\breve{\mathbf{v}}}_{r0}^{iT} + \varepsilon\bar{\breve{\mathbf{v}}}_{r1}^{iT}\right\}\tilde{\mathbf{D}}_f(\lambda_{r0}^i)\breve{\mathbf{u}}_{r1}^i}{\left\{\bar{\breve{\mathbf{v}}}_{r0}^{iT} + \varepsilon\bar{\breve{\mathbf{v}}}_{r1}^{iT}\right\}\tilde{\mathbf{D}}_f'(\lambda_{r0}^i)\breve{\mathbf{u}}_{r0}^i} \cong \frac{\varepsilon\bar{\breve{\mathbf{v}}}_{r1}^{iT}\tilde{\mathbf{D}}_f(\lambda_{r0}^i)\breve{\mathbf{u}}_{r1}^i}{\bar{\breve{\mathbf{v}}}_{r0}^{iT}\tilde{\mathbf{D}}_f'(\lambda_{r0}^i)\breve{\mathbf{u}}_{r0}^i} \sim 0, \quad \text{(A.10)}$$

$$\mathbf{u}_{r1}^i = -\mathbf{D}_f^{-1}(\lambda_{r0}^i)\tilde{\mathbf{D}}_r(\lambda_{r0}^i)\breve{\mathbf{u}}_{r0}^i = -\mathbf{D}_f^{-1}(\lambda_{r0}^i)\tilde{\mathbf{D}}_r(\lambda_{r0}^i)\bar{\mathbf{u}}_{-r0}^i. \quad \text{(A.11)}$$

References

[1] J.A. Richards, *Analysis of Periodically Time-Varying Systems*, Springer, New York, 1983.

[2] H. Irretier, Mathematical foundations of experimental modal analysis in rotor dynamics, *Mechanical Systems and Signal Processing* 13 (2) (1999) 183–191.

[3] G. Genta, Whirling of unsymmetrical rotors: a finite element approach based on complex coordinates, *Journal of Sound and Vibration* 124 (1) (1988) 27–53.

[4] C.W. Lee, *Vibration Analysis of Rotors*, Kluwer Academic Publishers, Dordrecht, 1993.

[5] Y.G. Jei, C.W. Lee, Modal analysis of continuous asymmetric rotor-bearing systems, *Journal of Sound and Vibration* 152 (2) (1992) 245–262.

[6] C.W. Lee, K.S. Kwon, Identification of rotating asymmetry in rotating machines by using reverse directional frequency response functions, *Proceedings of Institution of Mechanical Engineers* 215 (Part C) (2001) 1053–1063.

[7] C.W. Lee, S.K. Lee, An efficient complex modal testing theory for asymmetric rotor systems: use of unidirectional excitation method, *Journal of Sound and Vibration* 206 (3) (1996) 327–338.

[8] C.Y. Joh, C.W. Lee, Use of dFRFs for diagnosis of asymmetric/anisotopic properties in rotor-bearing system, *Journal of Vibration and Acoustics* 118 (1996) 64–69.

[9] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[10] A. Dimarogonas, Vibration of cracked structures: a state of the art review, *Engineering Fracture Mechanics* 55 (5) (1996) 831–857.

[11] S. Suresh, *Fatigue of Materials*, Cambridge University Press, Cambridge, 1991.

[12] C.W. Lee, Why we need to install exciters in rotating machines: benefits and prospects, *First International Symposium on Stability Control of Rotating Machinery (ISCORMA-1)*, South Lake Tahoe, Nevada, 2001.

[13] C.A. Papadopoulos, A.D. Dimarogonas, Coupled longitudinal and bending vibrations of a rotating shaft with an open crack, *Journal of Sound and Vibration* 117 (1) (1987) 81–93.

[14] H. Tada, P. Paris, G. Irwin, *The Stress Analysis of Cracks Handbook*, second ed., Research Cooperation, St. Louis, 1985.